$$0 \to H^1(\Pi_X/\ldots(G_K), \mathcal{O}_\Theta \otimes \mathbb{Z}/l) \to H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \mathcal{O}_\Theta \otimes \mathbb{Z}/l) \to H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \mathcal{O}_\Theta \otimes \mathbb{Z}/l)$$

$$0 \qquad \ddot{\eta}^\Theta \mapsto 0 \quad \text{(exact)}$$

$\rightsquigarrow$ Image of $\ddot{\eta}^\Theta \in H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \mathcal{O}_\Theta)$ arises from $l\mathcal{O}_\Theta \hookrightarrow \mathcal{O}_\Theta$

well-def. up to $(\mathcal{O}_K^\times)^l$

$\exists\, \underline{\ddot{\eta}}^\Theta \in H^1(\Pi^{\text{tp}}_{\ddot{Y}}, l\mathcal{O}_\Theta)$

well-def up to $\mathcal{O}_K^\times$-mult

"l-th root of the étale theta fact"

$\underline{\ddot{\eta}}^{\Theta, l\mathbb{Z}\times\mu_2}$: $\Pi^{\text{tp}}_{\underline{\ddot{X}}} / \Pi^{\text{tp}}_{\ddot{Y}} \cong l\mathbb{Z}\times\mu_2$-orbit of $\underline{\ddot{\eta}}^\Theta$

Def 7.12 ([EtTh, Def 2.7]) We call $\underline{\ddot{\eta}}^{\Theta, l\mathbb{Z}\times\mu_2}$ of standard type

$\overset{\text{def}}{\iff}$ $\ddot{\eta}^{\Theta, \mathbb{Z}\times\mu_2}$: of std type

$$
\begin{tikzcd}
 & H^1(s^{\mathrm{Assy}(3)}(G_k), \Delta_\Theta \otimes \mathbb{Z}/\ell) & & \\
0 \arrow[r] & H^1(\overline{\Pi}_X, \Delta_\Theta \otimes \mathbb{Z}/\ell) \arrow[u] & & \\
0 \arrow[r] & H^1(\overline{\Pi}_X / s^{\mathrm{Assy}(3)}(G_k), \Delta_\Theta \otimes \mathbb{Z}/\ell) \arrow[u] \arrow[r] & H^1(\Pi^{\mathrm{tp}}_{\ddot{Y}}, \Delta_\Theta \otimes \mathbb{Z}/\ell) \arrow[r] & H^1(\Pi^{\mathrm{tp}}_{\underline{\ddot{Y}}}, \Delta_\Theta \otimes \mathbb{Z}/\ell) \\
 & 0 \arrow[u] & &
\end{tikzcd}
$$

(exact)

$\ddot{\eta}^\Theta \mapsto 0$

$\ddot{\eta}^\Theta = s^{\mathrm{Assy}(3)}(G_k)$

(exact)

$$s^{\mathrm{Assy}(3)}(G_k) \ni s^{\mathrm{Assy}(3)}(g) \longmapsto s^{A(3)}(g)\left(s^{A(3)}\left(\overline{s^{A(3)}(g)}\right)\right)^{-1} = s^{A(3)}(g)\left(s^{A(3)}(g)\right)^{-1} = 1$$

$\rightsquigarrow$ Image of $\ddot{\eta}^\Theta \in H^1(\Pi^{\mathrm{tp}}_{\underline{\ddot{Y}}}, \Delta_\Theta)$ arises from $\ell\Delta_\Theta \hookrightarrow \Delta_\Theta$

$\exists\ \ddot{\eta}^\Theta \in H^1(\Pi^{\mathrm{tp}}_{\ddot{Y}}, \ell\Delta_\Theta)$

(1). γ : preserves the property that $\eta^{\Theta, \ell\mathbb{Z}, \mu_\ell}_{\underline{\underline{X}}}$ is of std type

det's this collection of classes up to a μ_ℓ-mult.

(2). Assume cusps of $\underline{\underline{X}}$ are rational / K, res. char. of $K \neq \ell$, $\mu_\ell \subset K$

$\Rightarrow$ $\{\pm 1\}$-str. of Prop 7.9 (3) det's $\mu_{2\ell}$-str. at the decomp. gps of the cusps of $\underline{\underline{X}}$

η^{Θ}

compatibly com. with the & preserved by γ.

Prop 7.9 & Cor 7.10

<u>Cor 7.14</u> ([EtTh, Cor 2.9])

Assume $\mu_\ell \subset K$

cusps of $\underline{\underline{X}}$ labelling labelling

nized. comp. $\exists$ by $\mathbb{Z}$

$$\{\text{Cusps of } \underline{\underline{X}}\} / \mathrm{Aut}_K(\underline{\underline{X}}) \xrightarrow{\sim} (\mathbb{Z}/\ell\mathbb{Z})/\{\pm 1\}$$

& preserved by $\forall \gamma : \Pi^{tp}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{tp}_{\underline{\underline{X}}}$ of tp. gps.

$\underline{\underline{X}}$

μ_ℓ

$\underline{X}$

X

labelling

labelled

Cor 7.13 (Const. Mult. Rig. of ℓ-th Root of the Etale Theta Fn. [EtTh, Cor 2.8])

$\underline{\underline{X}}$ (resp. $^{\dagger}\underline{\underline{X}}$) over K (resp. $^{\dagger}K$) $^{fin}/\mathbb{Q}_p$ as before

$^{\dagger}(-)$ $\quad \gamma : \Pi^{top}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{top}_{^{\dagger}\underline{\underline{X}}}$ isom. of top. gps

(1). γ : preserves the property that $\ddot{\eta}^{\Theta, \ell\mathbb{Z} \times \mu_2}$ is of std type

det's this collection of classes up to a μ_ℓ-mult.

(2). Assume cusps of $\underline{\underline{X}}$ are rational $/K$, res. char. of $K \neq \ell$, $\mu_\ell \subset K$

$\Rightarrow$ $\{\pm 1\}$-str. of Prop 7.9 (3) det's $\mu_{2\ell}$-str. at the decomp. gps of the cusps of $\underline{\underline{X}}$

η^{Θ}

compatibly com. with the ℓ preserved by γ.

Pf Prop 7.9 & Cor 7.10

Cor 7.14 ([EtTh, Cor 2.9])

Assume $\mu_\ell \subset K$

cusps of $\underline{\underline{X}}$ labelling labelling

mod. cusp. by $\mathbb{Z}$

$\{\text{Cusps of } \underline{\underline{X}}\} / \mathrm{Aut}_K(\underline{\underline{X}}) \xrightarrow{\sim} (\mathbb{Z}/\ell\mathbb{Z})/\{\pm 1\}$

& preserved by $\forall \gamma : \Pi^{top}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{top}_{\underline{\underline{X}}}$ of top. gps.

Aiscrete rig.

ell. cusp'ation

trivially

$\{\underline{q}^{j^2}\} \longrightarrow \underline{q}$ $\quad \underline{q}\,\Theta \subset \underline{q}^{j^2}\Theta$ $\quad \rightsquigarrow 0 \le -|\mathrm{ht}| + (\mathrm{indet})$

$|1+\epsilon|(\log\text{-diff} + \log\text{-cd})$

$N > 1$ $\quad \lambda \to 0$ $\quad \rightsquigarrow |\mathrm{ht}| \lesssim |1+\epsilon|(\cdots)$

① $\{\underline{q}^{j^2}\} \longrightarrow \underline{q}^N$? $\rightsquigarrow N \cdot 0 \le -|\mathrm{ht}| + (\mathrm{indet})$

works

It worked

② $\{(\underline{q}^{j^2})^N\} \longrightarrow \underline{q}$? $\rightsquigarrow 0 \le -N|\mathrm{ht}| + (\mathrm{indet})$

$\rightsquigarrow |\mathrm{ht}| \le \frac{1}{N}|1+\epsilon|(\log\text{-diff} + \log\text{-cd})$

does not work

$\rightsquigarrow$ contradiction analytic number theory

calculation in HA

$\sum_j j^2(\ ) \approx \frac{\ell^2}{24}(\cdot)$

$\|$

$\sum_j j(\ ) \approx \frac{\ell^2}{24}(\ )$

miracle equality

$\mathbb{M}^N \subset \Pi$

j (conj.) mult. rig.

[EtTh, Rem 5.12.5]

$[\ ,\ ]$ quadraticity $\to$ ~~mono-theta cycl. rig~~

[EtTh, Rem 2.19.3]

$\mathbb{M}^N$ $\quad \mathbb{M}^N$ $\quad N > 1$

$\mathcal{L}^{\otimes N} \not\cong \mathcal{L} \rightsquigarrow (\ )^{\otimes N}$

$\mathbb{M}$-cycl. rig.

$\to 0$

log-Kummer

[IUTchII, Rem 2.3.3]

labelling

labelled

top. gps.

the cusps

ell

ell

§7.4 Three Rigidities of Mono-Theta Environment

	mono-theta env. ς^{Θ}	bi-theta env. $\varsigma^{\Theta}, \varsigma^{\ell}$
cycl. rig.	delicately ○ quadraticity [,]	trivially ○
const. mult. rig.	delicately ○ ell. cusp'ation	trivially ○
Discrete rig.	○	X

(n 2.8])

μ_ℓ-mult.

ℓ, $\mu_\ell \subset K$

gps of the cusps of $\underline{X}$

by γ

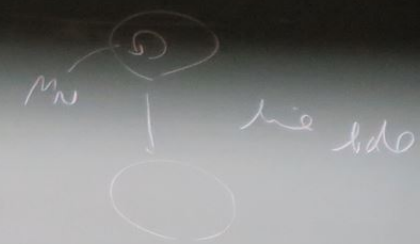
$q\theta \leftarrow q^{j^2}\theta$

$\{q^{j^2}\} \to q$

$N>1$


SEIKO
12 1 2 3 4 5 6 7 8 9 10 11

Def. 7.15 $N \geq 1$ $\quad \Pi_{\mu_N,K} := \mu_N \rtimes G_K$



For a top. gp Π w/ surj. $\Pi \twoheadrightarrow G_K$

put $\Pi[\mu_N] := \Pi \times_{G_K} \Pi_{\mu_N,K}$, $\quad \Delta[\mu_N] := \ker(\Pi[\mu_N] \twoheadrightarrow G_K)$

$$= \Delta \times \mu_N$$

$(\Delta := \ker(\Pi \twoheadrightarrow G_K))$

cyclotomic envelope of $\Pi \twoheadrightarrow G_K$

$\mu_N(\Pi[\mu_N]) := \ker(\Pi[\mu_N] \twoheadrightarrow \Pi)$

(mod N) cyclotome of the cycl. env. $\Pi[\mu_N]$

taut. sect. $G_K \longrightarrow \Pi_{\mu_N,K}$ of $\Pi_{\mu_N,K} \twoheadrightarrow G_K$

$\rightsquigarrow s^{alg}_{\Pi} : \Pi \longrightarrow \Pi[\mu_N]$

mod N tautological section

For any obj w/ $\Pi[\mu_N]$-conj. action, we call a μ_N-orbit a μ_N-conj. class

Lem 7.16 ([EtTh, Prop 2.11])

$\Pi \twoheadrightarrow G_K$ (resp. ${}^{\dagger}\Pi \twoheadrightarrow G_{{}^{\dagger}K}$) :

open subgp of the temp. or prof. fund. gp of hyperb. curve / a fin. ext'n K (resp. ${}^{\dagger}K$) / $\mathbb{Q}_p$

$\Delta := \ker(\Pi \twoheadrightarrow G_K)$ (resp. ${}^{\dagger}\Delta := \ker({}^{\dagger}\Pi \twoheadrightarrow G_{{}^{\dagger}K})$)

(1). $\ker(\Delta[\mu_N] \twoheadrightarrow \Delta)$ (resp. ${}^{\dagger}\Delta[\mu_N] \twoheadrightarrow {}^{\dagger}\Delta$) is equal to the center of $\Delta[\mu_N]$ (resp. ${}^{\dagger}\Delta[\mu_N]$).

(resp. $^\dagger\Delta = \ker(^\dagger\Pi \twoheadrightarrow G_k)$)

(1). $\ker(\Delta[\mu_N] \twoheadrightarrow \Delta)$ (resp. $^\dagger\Delta[\mu_N] \twoheadrightarrow {}^\dagger\Delta$)

is equal to the center of $\Delta[\mu_N]$ (resp. $^\dagger\Delta[\mu_N]$).

In particular, any isom $\Delta[\mu_N] \xrightarrow{\sim} {}^\dagger\Delta[\mu_N]$

is compat w/ $\Delta[\mu_N] \twoheadrightarrow \Delta$, $^\dagger\Delta[\mu_N] \twoheadrightarrow {}^\dagger\Delta$

(2). $\ker(\Pi[\mu_N] \twoheadrightarrow \Pi)$ (resp. $^\dagger\Pi[\mu_N] \twoheadrightarrow {}^\dagger\Pi$)

is equal to the union of the centers of the open subgps of $\Pi[\mu_N]$ (resp. $^\dagger\Pi[\mu_N]$)

In particular, any isom $\Pi[\mu_N] \xrightarrow{\sim} {}^\dagger\Pi[\mu_N]$

is compat w/ $\Pi[\mu_N] \twoheadrightarrow \Pi$

$^\dagger\Pi[\mu_N] \twoheadrightarrow {}^\dagger\Pi$

($\because$ slimness (Prop 2.7 (2a)(2b))

or

temp-slim (Prop 6.4))

are

$(G\dagger\kappa)$

<u>Prop. 7.17</u> ([EtTh, Prop 2.12])

(1). $\ker\left((\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta} \to (\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\mathrm{ell}} \right) = l\Delta_{\Theta} \overset{\text{we have an incl.}}{\subset} \left[(\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta}, (\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta} \right]$

(2). We have an equality

$$\left[(\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta}(\mu_N), (\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta}(\mu_N) \right] \cap (l\Delta_{\Theta})(\mu_N)$$

$$= \mathrm{Im}\left(s^{\mathrm{alg}}_{(\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta}}\Big|_{l\Delta_{\Theta}} : l\Delta_{\Theta} \to (\Delta^{\mathrm{tp}}_{\underline{\underline{X}}})^{\Theta}(\mu_N) \right)$$

s^{Θ}; s^{alg}

$\ker(|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})| \to |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\mathrm{ell}}) = l\triangle_\Theta \subset [|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}, |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}]$

(2). We have an equality

$$[|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}(\mu_N), |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}(\mu_N)] \cap (l\triangle_\Theta)(\mu_N)$$
$$= \mathrm{Im}(s^{\mathrm{alg}}_{|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}}|_{l\triangle_\Theta} : l\triangle_\Theta \to |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}(\mu_N))$$

s^{Θ} s^{alg}

$s^{\mathrm{alg}}_{|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}}|_{l\triangle_\Theta}$

restr. to $l\triangle_\Theta$

$s^{\mathrm{alg}}_{|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}} : |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta} \to |(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}(\mu_N)$

mod N tamb. section

(!) (1). str. of Heisenberg gp $|(\triangle^{\mathrm{trp}}_{\underline{\underline{X}}})|^{\Theta}$

(2) $\Leftarrow$ (1)

Rm 7.17.1

does not work

$\underline{\underline{X}}$

$\mathrm{Im}(s^{\mathrm{alg}})$

$\underline{X}$

$\ker({}^\dagger\Pi \to G_{{}^\dagger K})$

${}^\dagger\triangle$

$\mathrm{ver}, {}^\dagger\triangle(\mu_N)$

$\to \triangle, {}^\dagger\triangle(\mu_N) \to {}^\dagger\triangle$

$\to {}^\dagger\Pi$

center of the open

${}^\dagger\Pi(\mu_N)$

${}^\dagger\Pi(\mu_N)$

$\Pi(\mu_N) \to \Pi$

$$s^{alg}_{\underline{\underline{Y}}} : \Pi^{top}_{\underline{\underline{Y}}} \xrightarrow{s^{alg}_{\Pi^{top}_{\underline{\underline{Y}}}}} \Pi^{top}_{\underline{\underline{Y}}}[\mu_N] \hookrightarrow \Pi^{top}_{\underline{Y}}[\mu_N]$$

the composite

mod N algebraic section

$$\text{Take the composite } \eta : \Pi^{top}_{\underline{Y}} \longrightarrow \overset{\text{mod } N \text{ red.}}{l\Delta_\Theta \otimes \mathbb{Z}/N} \overset{\text{(cycl. thory)}}{\cong} \mu_N$$

$$\eta^{\Theta, l\underline{\mathbb{Z}} \times \mu_2} \subset H^1(\Pi^{top}_{\underline{Y}}, l\Delta_\Theta)$$

any elt in η

1-cocycle

Put $\varsigma^{\Theta}_{\underline{\underline{Y}}} := \eta^{-1} \cdot \varsigma^{alg}_{\underline{\underline{Y}}} : \Pi^{tp}_{\underline{\underline{Y}}} \longrightarrow \Pi^{tp}_{\underline{\underline{Y}}}[\mu_N]$

mod N theta section

Note $\varsigma^{\Theta}_{\underline{\underline{Y}}}$: homomorphic.

the natural outer action

$$\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \cong \Pi^{tp}_{\underline{\underline{X}}} / \Pi^{tp}_{\underline{\underline{Y}}} \cong \Pi^{tp}_{\underline{\underline{X}}}[\mu_N] / \Pi^{tp}_{\underline{\underline{Y}}}[\mu_N] \hookrightarrow \mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_N])$$

$\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \overset{\text{out.}}{\curvearrowright} \Pi^{tp}_{\underline{\underline{Y}}}[\mu_N]$ fixes $\mathrm{Im}(\varsigma^{alg}_{\underline{\underline{Y}}} : \Pi^{tp}_{\underline{\underline{Y}}} \to \Pi^{tp}_{\underline{\underline{Y}}}[\mu_N])$

up to conj. by μ_N

($\varsigma^{alg}_{\underline{\underline{Y}}}$ extends $\Pi^{tp}_{\underline{\underline{X}}} \to \Pi^{tp}_{\underline{\underline{X}}}[\mu_N]$)

$\mathrm{Gal}(\underline{Y}/\underline{X}) \cong \Pi_{\underline{X}}^{\mathrm{tp}}[\mu_n]/\Pi_{\underline{Y}}^{\mathrm{tp}}[\mu_n] \to \mathrm{Out}(\Pi_{\underline{Y}}^{\mathrm{tp}}[\mu_n])$

$\mathrm{Gal}(\underline{Y}/\underline{X}) \overset{\mathrm{out.}}{\curvearrowright} \Pi_{\underline{Y}}^{\mathrm{tp}}[\mu_n]$ fixes $\mathrm{Im}(s_{\underline{\ddot{Y}}}^{\mathrm{alg}} : \Pi_{\underline{\ddot{Y}}}^{\mathrm{tp}} \to \Pi_{\underline{\ddot{Y}}}^{\mathrm{tp}}[\mu_n])$ up to conj. by μ_n

($s_{\underline{Y}}^{\mathrm{alg}}$ extends $\Pi_{\underline{X}}^{\mathrm{tp}} \to \Pi_{\underline{X}}^{\mathrm{tp}}[\mu_n]$)

$\leadsto s_{\underline{\ddot{Y}}}^{\Theta}$ up to $\Pi_{\underline{X}}^{\mathrm{tp}}[\mu_n]$-conj. is indep. of the choice of
cn. dt. of $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu}$

we have natural outer action

$$K^\times \to K^\times/(K^\times)^N \cong H^1(G_K, \mu_n) \hookrightarrow H^1(\Pi_{\underline{Y}}^{\mathrm{tp}}, \mu_n) \to \mathrm{Out}(\Pi_{\underline{Y}}^{\mathrm{tp}}[\mu_n])$$

$$s \mapsto \left(\begin{array}{l} s_{\Pi_{\underline{Y}}}^{\mathrm{alg}}(g)\, a \mapsto s(g)\, s_{\Pi_{\underline{Y}}}^{\mathrm{alg}}(g) \cdot a \\ g \in \Pi_{\underline{Y}}^{\mathrm{tp}},\ a \in \mu_n \end{array} \right)$$

$\mu_N) \hookrightarrow \mathrm{Out}(\Pi^{tp}_{\underline{Y}}(\mu_N))$

$\mathrm{Im}(s^{alg}_{\underline{Y}} : \Pi^{tp}_{\underline{Y}} \to \Pi^{tp}_{\underline{Y}}(\mu_N))$

by μ_N

$\ker\left((\Delta^{tp}_{\underline{X}})^{\Theta} \to (\Delta^{tp}_{\underline{X}})^{ell} \right) = l\Delta_{\Theta} \subset \left[(\Delta^{tp}_{\underline{X}})^{\Theta}, (\Delta^{tp}_{\underline{X}})^{\Theta} \right]$

we have an incl.

(2). We have an equality

$$\left[(\Delta^{tp}_{\underline{X}})^{\Theta}(\mu_N), (\Delta^{tp}_{\underline{X}})^{\Theta}(\mu_N) \right] \cap (l\Delta_{\Theta})(\mu_N)$$

$$= \mathrm{Im}\left(s^{alg}_{(\Delta^{tp}_{\underline{X}})^{\Theta}}\big|_{l\Delta_{\Theta}} : l\Delta_{\Theta} \to (\Delta^{tp}_{\underline{X}})^{\Theta}(\mu_N) \right)$$

s^{Θ}

s^{alg}

---

ce of

$\Theta, l \geq$

$\mu_N) \longrightarrow \mathrm{Out}(\Pi^{tp}_{\underline{Y}}(\mu_N))$

Ψ

$\longrightarrow \left(s^{alg}_{\Pi^{tp}_{\underline{Y}}}(g)\, a \longmapsto s(g)\, s^{alg}_{\Pi_{\underline{Y}}}(g) \cdot a \right)$

$s \in \Pi^{tp}_{\underline{Y}},\ a \in \mu_N$

Note also

$\forall$ elt $\in \mathrm{Im}(K^{\times}) := \mathrm{Im}(K^{\times} \to \mathrm{Out}(\Pi^{tp}_{\underline{Y}}(\mu_N)))$

lifts to an elt $\in \mathrm{Aut}(\Pi^{tp}_{\underline{Y}}(\mu_N))$

which induces the id autom. of both the quot. $\Pi^{tp}_{\underline{Y}}(\mu_N) \to \Pi^{tp}_{\underline{Y}}$ & the kernel of this quot.

Def 7.18 (Mono-Theta Environment [EtTh, Def. 2.13])

$$\mathcal{D}_{\underline{Y}} := \langle \mathrm{Im}(K^\times), \mathrm{Gal}(\underline{Y}/\underline{X}) \rangle \subset \mathrm{Out}(\Pi^{\mathrm{tmp}}_{\underline{Y}}[\mu_N])$$

$$\underbrace{\mathrm{Gal}(\underline{Y}/\underline{X})}_{\ell\mathbb{Z}}$$

(1) We call the following collection of data a mod N model mono-theta environment :

(1) We call the following collection of data a mod N
mono-theta environment :

- the top. gp $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_N]$
- the subgp $D_{\underline{\underline{Y}}}$ $(\subset \mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_N]))$, and
- the μ_N-conj. class of subgps in $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_N]$ det'd by the image of the theta section $s^{\Theta}_{\underline{\underline{Y}}}$

(2). We call any collection $\mathbb{M} = (\Pi, D_\Pi, s^{\Theta}_{\Pi})$ of the following data a mod N mono-theta environment

- A top. gp Π
- a subgp $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$
- a collection of subgps $S^{\flat}_{\Pi}$ of Π

s.t. $\exists\ \Pi \xrightarrow{\sim} \Pi^{tp}_{\underline{Y}}(\mu_N)$ of top. gps which maps $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$ to $\mathcal{D}_{\underline{Y}}$ & $S^{\flat}_{\Pi}$ to the μ_N-conj. class of subgps in $\Pi^{tp}_{\underline{Y}}(\mu_N)$ det'd by the image of $S^{\flat}_{\ddot{\underline{Y}}}$

(3). $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^\Theta_\Pi)$, ${}^\dagger\mathbb{M} = ({}^\dagger\Pi, \mathcal{D}_{{}^\dagger\Pi}, s^\Theta_{{}^\dagger\Pi})$: mono-theta env.

<u>isom of mod N mono-theta env.</u> $\mathbb{M} \xrightarrow{\sim} {}^\dagger\mathbb{M}$

Ao be an isom. of top. gps $\Pi \xrightarrow{\sim} {}^\dagger\Pi$

which maps $\mathcal{D}_\Pi$ to $\mathcal{D}_{{}^\dagger\Pi}$, s^Θ_Π to $s^\Theta_{{}^\dagger\Pi}$,

$\mathbb{M}$: mod N mono-theta env. $\mathbb{M} = (\Pi, \mathcal{D}_\Pi, s^\Theta_\Pi)$

${}^\dagger\mathbb{M}$: mod M : :

M|N <u>a hom. of mono-theta env.</u> $\mathbb{M} \to {}^\dagger\mathbb{M}$

Ao be an isom $\underbrace{\mathbb{M}_M}_{\text{mod M mono-theta env. induced by } \mathbb{M}} \xrightarrow{\sim} {}^\dagger\mathbb{M}$

Rem 7.18. We can consider mod N bi-theta env $\mathbb{B} = (\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi}, s^{\mathcal{D}_\Upsilon}_{\Pi})$

Lem 7.19 ([EtTh, Prop 2.14])

(1). We have the following gp-thic characterization of the image of the tautological section of $(l\Delta_\Theta)(\mu_N) \to l\Delta_\Theta$ as the following subgp of $(\Delta^{+ty}_{\underline{Y}})^\Theta(\mu_N)$:

$$(l\Delta_\Theta)[\mu_N] \cap \left\{ \gamma(a) a^{-1} \in (\Delta^{+ty}_{\underline{Y}})^\Theta(\mu_N) \;\middle|\; \begin{array}{l} a \in (\Delta^{+ty}_{\underline{Y}})^\Theta(\mu_N), \\ \gamma \in \mathrm{Aut}(\Pi^{+ty}_{\underline{Y}}(\mu_N)) \text{ s.t. } ⊛ \end{array} \right\}$$

⊛ : the image of γ in $\mathrm{Out}(\Pi^{+ty}_{\underline{Y}}(\mu_N))$ belongs to $\mathcal{D}_{\underline{Y}}$

$\mathcal{D}_{\underline{Y}}$ ... by ... $s^{\Theta}_{\underline{Y}}$ ... env.

by

$s^\Theta_{\underline{Y}}$

... characterization of the image of the tors. section of $(l\Delta_\Theta)[\mu_N] \to l\Delta_\Theta$ as the following subgroup of $(\Delta^{\text{tp}}_{\underline{Y}})^\Theta[\mu_N]$:

$$(l\Delta_\Theta)[\mu_N] \cap \left\{ \gamma(a)a^{-1} \in (\Delta^{\text{tp}}_{\underline{Y}})^\Theta[\mu_N] \;\middle|\; \begin{array}{l} a \in (\Delta^{\text{tp}}_{\underline{Y}})^\Theta[\mu_N], \\ \gamma \in \mathrm{Aut}(\Pi^{\text{tp}}_{\underline{Y}}[\mu_N]) \text{ s.t. } \circledast \end{array} \right\}$$

env.

$\circledast$: the image of γ in $\mathrm{Out}(\Pi^{\text{tp}}_{\underline{Y}}[\mu_N])$ belongs to $\mathcal{D}_{\underline{Y}}$,

and γ induces the identity on the quot. $\Pi^{\text{tp}}_{\underline{Y}}[\mu_N] \twoheadrightarrow \Pi^{\text{tp}}_{\underline{Y}} \twoheadrightarrow G_K$.

(2). $t^\Theta_{\underline{Y}} : \Pi^{\text{tp}}_{\underline{Y}} \to \Pi^{\text{tp}}_{\underline{Y}}[\mu_N]$ a section obtained as a conj. of $s^\Theta_{\underline{Y}}$

rel. to the actions of $K^\times$ and $l\mathbb{Z}$

Put $\delta := (s^\Theta_{\underline{Y}})^{-1} t^\Theta_{\underline{Y}}$: 1-cocycle of $\Pi^{\text{tp}}_{\underline{Y}}$ valued in μ_N

$\alpha_\delta \in \mathrm{Aut}(\Pi^{\text{tp}}_{\underline{Y}}[\mu_N])$: the autom. given by $s^{\text{alg}}_{\Pi^{\text{tp}}_{\underline{Y}}}(g)a \mapsto \delta(g)\, s^{\text{alg}}_{\Pi^{\text{tp}}_{\underline{Y}}}(g)\, a$ $\quad (g \in \Pi^{\text{tp}}_{\underline{Y}},\ a \in \mu_N)$

which induces the id. aut. on both the quot.

$\Pi^{\text{tp}}_{\underline{Y}}[\mu_N] \twoheadrightarrow \Pi^{\text{tp}}_{\underline{Y}}$ & the kernel of this quot.

induced by M

Then "α_δ" extends to an autom. $\alpha_\delta \in \mathrm{Aut}(\Pi^{\mathrm{tpd}}_{\underline{Y}}[\mu_N])$

which induces an id. aut. on both the quot. $\Pi^{\mathrm{tpd}}_{\underline{Y}}[\mu_N] \twoheadrightarrow \Pi^{\mathrm{tpd}}_{\underline{Y}}$.

The conj. by α_δ maps $s^{\Theta}_{\underline{Y}}$ to $t^{\Theta}_{\underline{Y}}$ & its kernel

& preserves the subgp $\mathcal{D}_{\underline{Y}}$ $(\subset \mathrm{Out}(\Pi^{\mathrm{tpd}}_{\underline{Y}}[\mu_N]))$

(3). $\mathbb{M} = (\Pi^{\mathrm{tpd}}_{\underline{Y}}[\mu_N], \mathcal{D}_{\underline{Y}}, s^{\Theta}_{\underline{Y}})$: the mod N model mono-theta env.

$\forall$ autom. of $\mathbb{M}$ induces an autom of $\Pi^{\mathrm{tpd}}_{\underline{Y}}$ (by Lem 7.16(2))

$\rightsquigarrow$ autom. of $\Pi^{\mathrm{tpd}}_{\underline{X}} = \mathrm{Aut}(\Pi^{\mathrm{tpd}}_{\underline{Y}}) \overset{\mathrm{out}}{\rtimes} \mathrm{Im}(\mathcal{D}_{\underline{Y}} \to \mathrm{Out}(\Pi^{\mathrm{tpd}}_{\underline{Y}}))$

$= \mathrm{Aut}(\Pi^{\mathrm{tpd}}_{\underline{Y}}) \times_{\mathrm{Out}(\Pi^{\mathrm{tpd}}_{\underline{Y}})} \mathrm{Im}(\mathcal{D}_{\underline{Y}} \to \mathrm{Out}(\Pi^{\mathrm{tpd}}_{\underline{Y}}))$

(3). $\mathbb{M} = (\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N), D_{\underline{Y}}, s^{\flat}_{\underline{Y}})$: the mod N model mono-theta env.

$\Rightarrow$ autom. of $\mathbb{M}$ induces an autom of $\Pi^{\mathrm{tp}}_{\underline{Y}}$ (by Lem 7.16 (2))

$\rightsquigarrow$ autom. of $\Pi^{\mathrm{tp}}_{\underline{X}} = \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\underline{Y}}) \overset{\mathrm{out}}{\rtimes} \mathrm{Im}(D_{\underline{Y}} \to \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}))$

$= \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\underline{Y}}) \times_{\mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}})} \mathrm{Im}(D_{\underline{Y}} \to \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}))$

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next week

1F

Room

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also induces $\longrightarrow$ an autom. of the set of cusps of $\underline{Y}$

w.r.t. the labelling by $\underline{\mathbb{Z}}$ on these cusps,

this induces an autom of $\underline{\mathbb{Z}}$ given by $l\underline{\mathbb{Z}} \rtimes \{\pm 1\}$

$\rightsquigarrow \mathrm{Aut}(\mathbb{M}) \to l\underline{\mathbb{Z}} \rtimes \{\pm 1\}$

☺ (1). Take a lift $\gamma \in \mathrm{Aut}((\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N))$ of an elt $\in \mathrm{Im}(K^\times) \subset D_{\underline{Y}}$

s.t. satisfies ⊛

$\gamma = \gamma_1 \gamma_2$ $\gamma_1 \in \mathrm{Inn}(\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N))$, $\gamma_2 \in \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N))$ s.t.

the image of γ_2 in $\mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N))$ in $\mathrm{Im}\begin{pmatrix} K^\times \to H^1(G_K, \mu_N) \\ \to H^1(\Pi^{\mathrm{tp}}_{\underline{Y}}, \mu_N) \\ \to \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N)) \end{pmatrix}$

& the autom induced by γ_2 on the quot.

$\Pi^{\mathrm{tp}}_{\underline{Y}}(\mu_N) \to \Pi^{\mathrm{tp}}_{\underline{Y}}$ & its ker are trivial

$$\begin{tikzcd}
H^1(G_K, \mu_N) \arrow[r] \arrow[rr, bend right, "0"'] & H^1(\Pi^{tp}_{\underline{\underline{Y}}}, \mu_N) \arrow[r] \arrow[rd, bend right, "0"'] & H^1(\Delta^{tp}_{\underline{\underline{Y}}}, \mu_N) \arrow[d] \\
 & & \mathrm{Out}(\Delta^{tp}_{\underline{\underline{Y}}}(\mu_N))
\end{tikzcd}$$

$\rightsquigarrow$ $\gamma_2 \curvearrowright \Delta^{tp}_{\underline{\underline{Y}}}(\mu_N)$ is inner

On the other hand, $\gamma_1 \curvearrowright G_K$ trivial ($\Leftarrow$ $\gamma_2 \curvearrowright G_K$ trivial & condition $\circledast$)

G_K: center-free (Prop 2.1)(1c)

$\rightsquigarrow \gamma_1 \in \mathrm{Inn}(\Pi^{tp}_{\underline{\underline{Y}}}(\mu_N))$ is in $\in \mathrm{Inn}(\Delta^{tp}_{\underline{\underline{Y}}}(\mu_N))$

$\rightsquigarrow \gamma = \gamma_1 \gamma_2 \curvearrowright \Delta^{tp}_{\underline{\underline{Y}}}(\mu_N)$ is also inner

$\rightsquigarrow$ the inner autom. induced by γ on $(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mu_N)$ is trivial (Prop 1.1(2))

$\left((\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mu_N) \cong l\underline{\mathbb{Z}} \times \hat{\mathbb{Z}}(1) \times \mu_N \right)$ abelian

π_1 center-free (Prop 2.1)(1c)

$\leadsto \gamma_1 \in \mathrm{Inn}(\Pi^{tp}_{\underline{Y}}(\mu_N))$ is in $\subset \mathrm{Inn}(\Delta^{tm}_{\underline{Y}}(\mu_N))$

$\leadsto \gamma = \gamma_1 \gamma_2$ on $\Delta^{tp}_{\underline{Y}}(\mu_N)$ is also inner

$(\Delta^{tp}_{\underline{Y}})^{\Theta}(\mu_N) \cong l\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}(1)(\mu_N)$ abelian $\leadsto$ the inner automorphism induced by γ on $(\Delta^{tp}_{\underline{Y}})^{\Theta}(\mu_N)$ is trivial $\overset{(1)}{=}$ Prop 1.1(2)

(2): similar

(3) $\Leftarrow$ (2) //

<u>Cor 7.20</u> (Group-Theoretic Reconstruction of Mono-Theta Environment) [EtTh, Cor 2.18]

$N \geq 1$, l: prime, $\underline{X}$ smooth log-curve of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$

$l, p > 2$, $\ddot{K} = K$, $/ K \supseteq \mathbb{Q}_p$

$\mathbb{M}_N$: the mod N model mono-theta env.

(1). ${}^{\dagger}\Pi^{tp}_{\underline{X}}$ top. gp. $\cong \Pi^{tp}_{\underline{X}}$

$\underbrace{}_{\text{gp th'c alg'm.}}$ subquot. ${}^{\dagger}\Pi^{tp}_{\underline{Y}}$, ${}^{\dagger}\Pi^{tp}_{\underline{\ddot{Y}}}$, ${}^{\dagger}G_K$, ${}^{\dagger}(l\Delta_{\Theta})$, ${}^{\dagger}(\Delta^{tp}_{\underline{X}})^{\Theta}$, ${}^{\dagger}(\Pi^{tp}_{\underline{X}})^{\Theta}$, ${}^{\dagger}(\Delta^{tm}_{\underline{Y}})^{\Theta}$, ${}^{\dagger}(\Pi^{tp}_{\underline{Y}})^{\Theta}$ of ${}^{\dagger}\Pi^{tp}_{\underline{X}}$

$\ddot{U}$ $\underline{\ddot{Y}}$

$\downarrow$

U $\underline{Y}$

(2)

$\Pi \mapsto M$

(3)

$M \mapsto \Pi$

and • a collection of subgps of $^{\dagger}\Pi^{tp}_{\underline{\underline{X}}}$ for each elt $(\mathbb{Z}/\ell\mathbb{Z})/\{\pm 1\}$

s.t. any isom. $^{\dagger}\Pi^{tp}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{tp}_{\underline{\underline{X}}}$ maps

• the above subquots to the subquots

$\Pi^{tp}_{\underline{Y}}, \Pi^{tp}_{\underline{\underline{Y}}}, G_K, l\Delta_{\Theta}, (\Delta^{tp}_{\underline{\underline{X}}})^{\Theta}, (\Pi^{tp}_{\underline{\underline{X}}})^{\Theta}, (\Delta^{tp}_{\underline{Y}})^{\Theta}, (\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}$

of $\Pi^{tp}_{\underline{\underline{X}}}$ resp'ly

and • the above collection of subgps to the collection of cuspidal decomp. gps of $\Pi^{tp}_{\underline{\underline{X}}}$ det'd by the label in $(\mathbb{Z}/l\mathbb{Z})/\{\pm 1\}$

in a functorial manner w.r.t. isoms of top. gps.

(no need of any reference isom. to $\Pi^{tp}_{\underline{\underline{X}}}$)

• the above subquots to the subquots

$\Pi^{\dagger tp}_{\underline{Y}}$, $\Pi^{\dagger tp}_{\underline{\underline{Y}}}$, G_k, $\ell\Delta_\Theta$, $({}^\dagger\Delta^{tp}_{\underline{X}})^\Theta$, $(\Pi^{tp}_{\underline{X}})^\Theta$, $(\Delta^{tp}_{\underline{Y}})^\Theta$, $(\Pi^{tp}_{\underline{Y}})^\Theta$

of $\Pi^{tp}_{\underline{X}}$ resp'ly

and • the above collection of subgps to the collection of cuspidal decomp. gps of $\Pi^{tp}_{\underline{X}}$ lab'd by the label in $(\mathbb{Z}/\ell)/\{\pm1\}$

in a functorial manner w.r.t. isoms of top. gps.

[no need of any reference isom. to $\Pi^{tp}_{\underline{X}}$]

(2) ∃ gp th'c alg'm

$({}^{\prime\prime}\Pi \rightsquigarrow M^{\prime\prime})$ ${}^\dagger\Pi^{tp}_{\underline{X}} \rightsquigarrow {}^\dagger M = ({}^\dagger\Pi, \mathcal{O}_{{}^\dagger\Pi}, s^\Theta_{{}^\dagger\Pi})$

where ${}^\dagger\Pi := {}^\dagger\Pi^{tp}_{\underline{Y}} \times_{{}^\dagger G_k} (\ell\,{}^\dagger\Delta_\Theta \otimes \mathbb{Z}/n) \rtimes {}^\dagger G_k)$

up to isom in a functorial manner w.r.t. isom. of top. gps [no need of any ref. isom. to $\Pi^{tp}_{\underline{X}}$]

(See also [EtTh, Cor 2.18(iii)] for a stronger form)